Monotonicity of Multi-dimensional Limiting Process on Unstructured Grids

Jin Seok PARK [1], Sung-Hwan YOON [1] and Chongam KIM [1,2]

1) *School of Mechanical Aerospace Eng., Seoul National University, Seoul 151-742, KOREA*
2) *Institute of Advanced Aerospace Technology, School of Mechanical and Aerospace Eng., Seoul National University, Seoul, 151-742, KOREA*

Corresponding Author: Chongam KIM, chongam@snu.ac.kr

ABSTRACT

The present paper deals with the continuous work of extending multi-dimensional limiting process (MLP), which has been quite successfully proposed on two- and three-dimensional structured grids, onto the unstructured grids. The basic idea of the present limiting strategy is to control the distribution of both cell-centered and cell-vertex physical properties to mimic a multi-dimensional nature of flow physics, which can be formulated as so called the MLP condition. This condition satisfies the maximum principle, which ensures monotonicity, and numerical results show that MLP is effective to prevent unwanted oscillations as well as to capture multi-dimensional flow features accurately.

INTRODUCTION

High resolution scheme is one of the challengeable issues in hyperbolic conservation laws. Especially, discontinuities in the solution may lead spurious oscillations, which break down the numerical solution. For this reason, there have been many remarkable progresses on oscillation-free scheme, such as TVD or ENO, but most of them rely on the mathematical analysis of one-dimensional convection equation. Though this approach may work successfully in many cases, it is often insufficient or almost impossible to control oscillations near shock discontinuity in multi-dimensional flow.

In order to find out the suitable criterion to prevent such oscillations in multiple dimensions, the one-dimensional monotonic condition was extended to multi-dimensional flow situations and the multi-dimensional limiting process (MLP) was successfully developed. From the series of researches, it has been clearly demonstrated that the MLP limiting strategy possesses favorable characteristics, such as enhanced accuracy and convergence behavior in numerous inviscid and viscous computations on structured grids [1, 2]. Furthermore, the MLP limiting strategy can be readily extended on unstructured grids with some modifications [3].

In this work, we explore the monotonicity of MLP on unstructured grids. After introducing the basic concept and the implementation, it is shown that the proposed scheme satisfies the maximum principle. Various numerical tests are presented to verify the performance of the proposed method.

BASIC CONCEPT AND IMPLEMENTATION

In order to maintain multi-dimensional monotonicity, the present limiting strategy exploits the MLP condition, which is an extension of the one-dimensional monotonic condition. On the structured grids, the MLP condition restricts the physical properties on vertex as well as cell-center points. On vertex, a physical property is estimated by summing the monotonic variation of each coordinate direction, and then the vertex values are required to satisfy the following the MLP condition (Eq.(1)). On cell-center point, like conventional limiting schemes, it is controlled by limiting the cell-interface values.

$$\overline{q}_{neighbor}^{\min} \le \hat{q}_{vertex} \le \overline{q}_{neighbor}^{\min} , \tag{1}$$

where q is a state variable and $\hat{q}_{vertex}$ is the values at vertex. $\left(\overline{q}_{neighbor}^{\min}, \overline{q}_{neighbor}^{\max}\right)$ are the minimum and maximum of the cell-averaged values among the neighboring cells sharing this vertex.

On unstructured grids, there is no explicit basis direction, and thus it not feasible to obtain directional variations and to restrict cell interface values by adjusting these variations. To cope with this multi-dimensional nature, the interpolation stage starts from the MUSCL-type framework on unstructured grids, which can be written as follows.

$$q_j(\mathbf{x}) = \overline{q}_j + \phi \nabla \overline{q}_j \cdot \mathbf{r} , \tag{2}$$

where $\nabla \overline{q}_j$ is the gradient of the component on the cell T_j and ϕ is a slope limiter.

Applying the MLP condition into the MUSCL-type framework, the value at the vertex is limited considering all of the distributions around the vertex itself. The range of the limiter is expressed as follows.

$$\frac{\overline{q}_{neighbor}^{\min} - \overline{q}}{\nabla q \cdot \mathbf{r}_{vertex}} \le \phi \le \frac{\overline{q}_{neighbor}^{\max} - \overline{q}}{\nabla q \cdot \mathbf{r}_{vertex}} . \tag{3}$$

Within this range, MLP slope limiter is devised and the general formulation of it can be written as follows

$$\phi_{MLP} = \min \begin{cases} \Phi\left(r_i^{\max}\right) & if\ \nabla q \cdot \mathbf{r}_{v_i} > 0 \\ \Phi\left(r_i^{\min}\right) & if\ \nabla q \cdot \mathbf{r}_{v_i} < 0 \\ 1 & if\ q_i = q_A \end{cases} \tag{4}$$

where $r_i^{\min/\max} = (\hat{q}_{v_i,j}^{\min/\max} - \overline{q}_j) \big/ \nabla q \cdot \mathbf{r}_{v_i}$. For monotonicity, Φ should be in the range of $0 \le \Phi(r) \le \min(1, r)$. The immediate form of the characteristic limiting function Φ is to choose the upper bound of the limiting region. This limiter denotes as MLP-u1, which can be written as follows.

$$\Phi(r) = \min(1, r) \tag{5}$$

MLP-u1 is non-differentiable, which might have a potential to hamper the convergence of steady state problems. Adapting the modification by Venkatakrishnan of Barth's limiter [4], we also propose MLP-u2 limiter for steady state problem as follows.

$$\Phi\left(\frac{\Delta_+}{\Delta_-}\right) = \frac{1}{\Delta_-}\left[\frac{\left(\Delta_+^2 + \varepsilon^2\right)\Delta_- + 2\Delta_-^2\Delta_+}{\Delta_+^2 + 2\Delta_-^2 + \Delta_+\Delta_- + \varepsilon^2}\right], \tag{6}$$

where $\varepsilon^2 = (K\Delta x)^3$. The role of value ε is to distinguish a nearly smooth region from a fluctuating one. Like TVB or ELED limiters, it also plays a role of preventing clipping phenomenon.

SATISFACTION OF MAXIMUM PRINCIPLE

The effectiveness of the MLP condition is supported by the maximum principle, which is a complementary condition ensuring the monotonicity on multiple dimensions. For the convenience of illustration, this feature is proved on 2-D triangular meshes and the result is summarized in the following theorem.

Theorem. For a fully discrete finite volume scheme of hyperbolic conservation laws with a Lipschitz continuous flux function, if the linear reconstruction satisfies the MLP condition under a proper CFL restriction, then the scheme satisfies the maximum principle, i.e.,

$$\overline{q}_{j,neighbor}^{\min,n} \le \overline{q}_j^{n+1} \le \overline{q}_{j,neighbor}^{\max,n} . \tag{7}$$

The $\overline{q}_{j,neighbor}^{\min,n}$ and $\overline{q}_{j,neighbor}^{\max,n}$ are the minimum and maximum cell-averaged values among the neighborhood of the cell T_j, which shares at least a common point with the cell T_j.

Detail proof will be addressed in the presentation. While other limiters on unstructured grids, such as Barth's limiter, satisfy the maximum principle, the difference can be shown by comparing the stencil for the maximum principle (See Fig. 1). Since the allowable limiting range of these limiters essentially comes from Spekreijse's monotonic condition [5], the stencil of these limiters includes neighboring cells which only share an edge of the cell to be updated. Thus they are generally sensitive local mesh distribution and have a drawback not to capture multi-dimensional discontinuity accurately. However, as shown in Fig. 1(c), the MLP condition fully exploits all of the cell-averaged values sharing vertices, as well as edges, so MLP limiting is less sensitive to local mesh distribution and able to accurately detect discontinuities, especially near vertex point.

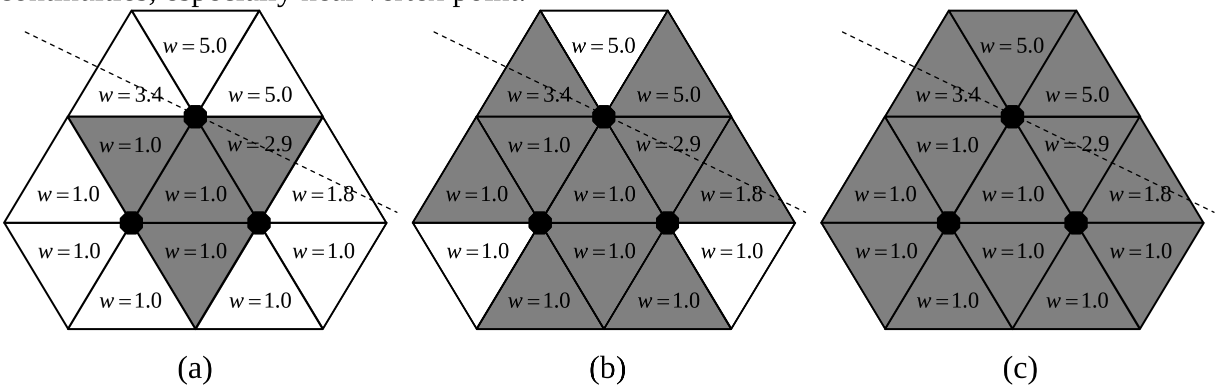


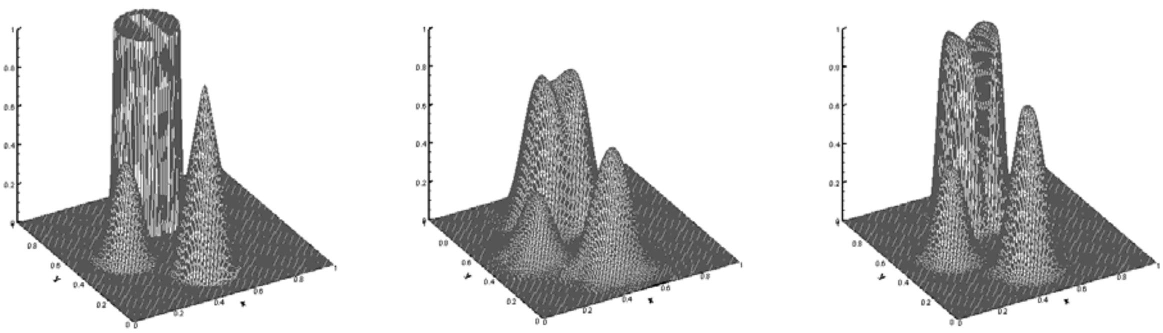
Figure 1. Comparison of stencil for each limiting strategy: (a) Maximum principle region by M. E. Hubbard, (b) Stencil of Barth's limiter, (c) MLP limiting region.

NUMERICAL RESULT

Solid Body Rotation

Solid body rotation problem is a good benchmark to assess the performance of numerical scheme where the flow is not constant. The wave velocity is $\mathbf{a} = \left(-\left(y-1/2\right), \left(x-1/2\right)\right)$ and computational domain is [0,1]x[0,1]. The initial profile consists of smooth hump, cone and slotted cylinder. Each shape lies within the circle of radius and the solution on the rest of region is initialized as zero.

Figure 2 shows the computed results after one revolution $(t = 2\pi)$ with *100x100x2* grid, whose triangular elements created by dividing uniform square elements along the diagonal. Comparing to the result of Barth's limiter, the result of MLP-u1 maintains the initial shape and does not produce unwanted oscillations around discontinuities.



(a) Initial profile (b) Barth's Limiter (c) MLP-u1

Figure 2. Numerical result of solid body rotation problem after one revolution

Shock tube problem

This test is performed to examine the capability to resolve various linear and non-linear waves on unstructured grids. Computational domain is [0,1]x[-0.05,0.05]x[-0.05,0.05] and the grid system is composed of *60,000* tetrahedral elements. RoeM scheme is applied as a numerical flux.

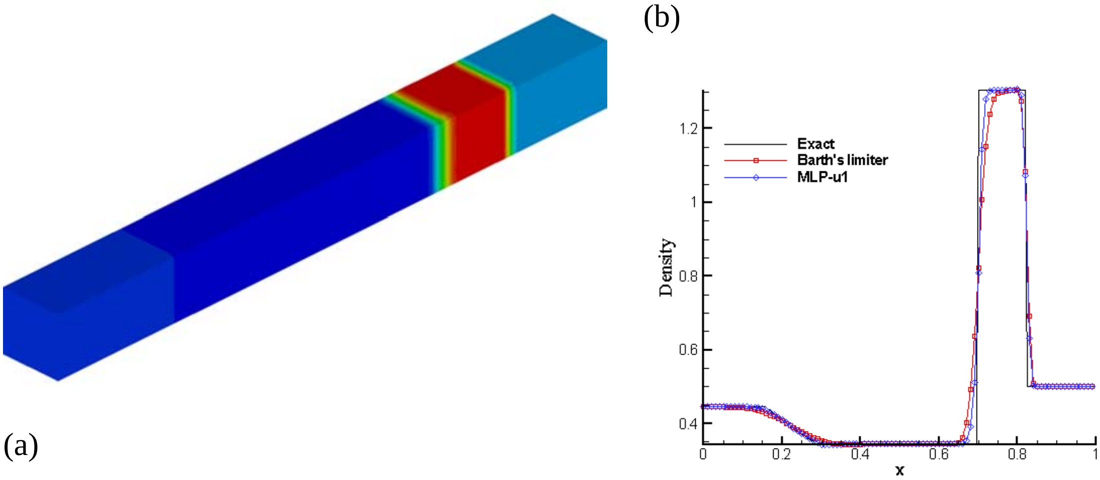


Figure 3. Density distributions of Lax problem: (a) on surface, (b) along the centerline

In Fig. 3, density distributions of Lax problem are compared at *t=0.13*. Though both limiters give monotone solution, one can see the better resolution of MLP limiting.

CONCLUSION

In this paper, a new robust and accurate limiting process for multi-dimensional hyperbolic conservations laws on unstructured grids is presented. The MLP condition on unstructured grids controls physical values by exploiting all of the neighboring cell-centered values. Consequently, the limiting stencil becomes optimally compact and wide enough to adequately capture multi-dimensional nature. Moreover, the satisfaction of maximum principle enforces multi-dimensional monotonicity. Numerical tests demonstrate the superior characteristics of the proposed limiting strategy over conventional limiter.

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